Rheology of Neck Formation in the Cold Drawing of Polymeric Fibers

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SYNOPSIS

An asymptotic formula has been obtained for the axial tension in nonlinear viscoelastic fibers undergoing inhomogeneous stretch. The formula, which is valid to within an error of the fourth order in diameter, expresses the tension per unit cross-sectional area in terms of the history of the local axial stretch and its first two derivatives with respect to distance along the fiber axis. The theory obtained by treating the formula as exact is consistent with thermodynamical principles, and permits computation of the stretch field resulting from a specified tensile loading history. Numerical results for creep under static load show that for an appropriate class of materials with slowly fading memory there is a range of applied loads for which an initially homogeneous deformation evolves into a well-defined neck whose edges, after a period of relatively quiescent incubation, advance rapidly along the fiber and in so doing transform moderately stretched material into highly stretched, i.e., drawn material. The calculated fiber profiles and the predicted dynamics of neck formation are in accord with familiar observations of neck formation in polymeric materials susceptible to cold drawing.

INTRODUCTION

In 1932, the phenomenon called "cold drawing" was described as follows by Carothers and Hill,[1] who clearly realized its eventual importance for the production of polymeric fibers and films of high strength and uniform properties:

> In connection with the formation of fibers the ω-polymers exhibit a rather spectacular phenomenon which we call cold drawing. If stress is gently applied to a cylindrical sample of the opaque, unoriented 3–16 ω-polyester at room temperature or at a slightly elevated temperature, instead of breaking apart, it separates into two sections joined by a thinner section of the transparent, oriented fiber. As pulling is continued this transparent section grows at the expense of the unoriented sections until the latter are completely exhausted. A remarkable feature of this phenomenon is the sharpness of the boundary at the junction between the transparent and opaque sections of the filament. During the drawing operation the shape of this boundary does not change; it merely advances through the opaque sections until the latter are exhausted. This operation can be carried out very rapidly and smoothly, and it leads to oriented fibers of uniform cross section.

In the subsequent literature the "transparent" or "thinner" section is referred to as the *neck.*

The materials in which cold drawing was first observed and is most frequently studied are semicrystalline polymers. Because cold drawing, accompanied by a stable neck of the type described by Carothers and Hill, has been observed under the nearly isothermal conditions that result when either low rates of stretching or sufficiently thin specimens are employed, adiabatic heating does not appear necessary for the phenomenon. We discuss below some results in a recently formulated theory[2] of the rheology of neck formation in isothermal cold drawing. To obtain the theory we treat a fiber as a three-dimensional body composed of an incompressible simple fluid[3] with slowly fading memory,[4,5] and we derive, for the limit of small fiber diameter, an asymptotic formula for the axial tension T resulting from inhomogeneous stretch. The formula gives T as a functional of the history of the axial stretch and its first two derivatives with respect to distance

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along the fiber axis. This formula and an associated asymptotic expression for the free energy per unit length are then employed as the basic constitutive relations of a unidimensional body, and the resulting model of a fiber with slowly fading memory is shown to be consistent with thermodynamical principles. The model has been employed to calculate the creep response of fibers under static loads. Numerical solutions of the evolution equation for the stretch in creep show that for an appropriate class of materials with long range memory there is a range of applied loads for which an initially homogeneous deformation evolves into a well-defined neck whose edges advance at high speed along the fiber and in so doing transform moderately stretched material into highly stretched (i.e., drawn) material. The calculated fiber profiles and the predicted dynamics of neck formation and growth are in good accord with familiar observations.

ON THE TENSION IN VISCOELASTIC FIBERS

To describe a motion of a three-dimensional body, one may write $\chi_{[t]}(\mathbf{x}, \zeta)$ for the place in space, at time ζ, of the material point that is at the place $\mathbf{x}$, at a time t which is interpreted as the present time. If one puts $\mathbf{F}_{[t]}(\zeta) = \nabla_{\mathbf{x}}\chi_{[t]}(\mathbf{x}, \zeta)$, then the symmetric positive-definite tensor $\mathbf{F}_{[t]}(\zeta)^T\mathbf{F}_{[t]}(\zeta)$ is the right Cauchy-Green tensor at time ζ, computed using the configuration at time t as the reference. Now, let $\mathbf{G}(\zeta) \doteq [\mathbf{F}_{[t]}(\zeta)^T\mathbf{F}_{[t]}(\zeta)]^{-1}$ and define the *history up to time t* of $\mathbf{G}$ to be the function $\mathbf{G}^t$ on $[0, \infty)$ for which

$$\mathbf{G}^t(s) = \mathbf{G}(t - s)$$
$$= \mathbf{F}_{[t]}(t-s)^{-1}\,[\mathbf{F}_{[t]}(t-s)^{-1}]^T \quad (s \geq 0) \qquad (1)$$

At each material point of a simple incompressible fluid, the Helmholtz free-energy density ψ (per unit volume) and the tensor $\mathbf{S}_e$, which is related to the stress tensor $\mathbf{S}$ of Cauchy by the equation

$$\mathbf{S} = -p\mathbf{1} + \mathbf{S}_e \qquad (2)$$

are given by constitutive equations of the form

$$\psi(t) = \mathscr{p}(\mathbf{G}^t) \qquad \mathbf{S}_e(t) = \mathscr{S}(\mathbf{G}^t) \qquad (3)$$

in which $\mathscr{p}$ and $\mathscr{S}$ are functionals that are isotropic in the sense that they obey the identities

$$\mathscr{p}(\mathbf{Q}\mathbf{G}^t\mathbf{Q}^T) = \mathscr{p}(\mathbf{G}^t)$$
$$\mathscr{S}(\mathbf{Q}\mathbf{G}^t\mathbf{Q}^T) = \mathbf{Q}\mathscr{S}(\mathbf{G}^t)\mathbf{Q}^T \qquad (4)$$

for each orthogonal tensor $\mathbf{Q}$, i.e., each tensor $\mathbf{Q}$ with $\mathbf{Q}\mathbf{Q}^T = \mathbf{1}$, the unit tensor. In eq. (2), $-p\mathbf{2}$ is an unspecified hydrostatic pressure; $\mathbf{S}_e$ is called the extra-stress tensor.

As we here confine attention to isothermal processes, the second law of thermodynamics is equivalent to the assertion that, if we put

$$\theta\sigma(t) = \mathscr{S}(\mathbf{G}^t)\cdot\mathbf{L} - \frac{d}{dt}\mathscr{p}(\mathbf{G}^t) \qquad (5)$$

with $\mathbf{L} = \mathbf{L}(t)$ the present value of the velocity gradient and θ the (constant) absolute temperature, then for all choices of $\mathbf{G}^t$ and $\mathbf{L}$ (subject to the constraint of incompressibility which requires that $\det \mathbf{G}^t = 1$ and $\operatorname{tr} \mathbf{L} = 0$), we must have

$$\sigma(t) \geq 0 \qquad (6)$$

because σ is the local rate of production of entropy. In the thermodynamics of materials with memory it is shown that, under appropriate smoothness assumptions for constitutive functionals, the requirement that $\sigma(t)$ never be negative implies that the free-energy functional $\mathscr{p}$ determines the stress functional $\mathscr{S}$ by a formula called the "generalized stress relation".[6]

The calculations we report here were done for a class of incompressible fluids for which $\mathscr{p}$ has the form

$$\psi(t) = \mathscr{p}(\mathbf{G}^t) = \sum_{j=1}^{n}\int_0^\infty K_j(s)H_j(\mathbf{G}^t(s))ds \qquad (7)$$

where each of the functions K_j is positive on $[0, \infty]$ with

$$K_j'(s) \leq 0 \qquad (s \geq 0) \qquad (8)$$

and is normalized so that

$$\int_0^\infty K_j(s)ds = 1 \qquad (9)$$

and each of the functions H_j obeys the relations

$$H_j(\mathbf{G}) \geq H_j(\mathbf{1}) = 0 \qquad (10)$$

[In eq. (8), $K_j'(s) = dK_j(s)/ds$.]

Application of the generalized stress relation to (7) yields

$$\mathscr{S}(\mathbf{G}^t) = 2\sum_{j=1}^{n}\int_0^{\infty} K_j(s)\mathbf{G}^t(s)\nabla H_j(\mathbf{G}^t(s))\,ds \tag{11}$$

where ∇H_j is the tensor-valued gradient of H_j. Materials obeying this last relation form a class of BKZ fluids.[7] For the example to be considered in detail below, the functions K_j and H_j are chosen such that, although the material has the symmetry of a simple fluid and obeys a weak hypothesis of fading memory, it does not have finite steady-state viscosity, for it cannot undergo either steady shearing flow or steady extension[8] at finite stress.

When $\mathscr{p}$ is as in (7) and $\mathscr{S}$ as in (11), eq. (5) implies the relation

$$\theta\sigma(t) = -\sum_{j=1}^{n}\int_0^{\infty} K_j'(s)H_j(\mathbf{G}^t(s))\,ds \tag{12}$$

and hence, by (8) and (10), the dissipation inequality (6) does hold here. It follows from $(4)_1$ that the functions H_j are isotropic and hence there are completely symmetric functions h_j of positive triples such that $H_j(\mathbf{G}) = h_j(\xi_1, \xi_2, \xi_3)$, where ξ_1, ξ_2, ξ_3 are the proper numbers of $\mathbf{G}$; thus, $(3)_1$ and (7) yield

$$\psi(t) = \mathscr{p}(\mathbf{G}^t) = \sum_{j=1}^{n}\int_0^{\infty} K_j(s)h_j(\xi_1^t(s), \xi_2^t(s), \xi_3^t(s))\,ds \tag{13}$$

where the $\xi_i^t(s)$, $i = 1, 2, 3$, are the proper numbers of $\mathbf{G}^t(s)$, and, by (10),

$$h_j(\xi_1^t(s), \xi_2^t(s), \xi_3^t(s)) \geq h_j(1, 1, 1) = 0 \tag{14}$$

It is further assumed that, for each positive number ξ,

$$(\xi - 1)[\partial_1 h_j(\xi^2, \xi^{-1}, \xi^{-1}) - \xi^{-3}\partial_2 h_j(\xi^2, \xi^{-1}, \xi^{-1})] \geq 0 \tag{15}$$

This last assumption is equivalent to the assertion that in the rapid homogeneous stretch of a previously undeformed circularly cylindrical bar the resulting axial tension T is positive for axial extension ($\lambda > 1$) and negative for axial compression ($0 < \lambda < 1$); an assumption that is clearly in accord with experience.

Consider now a body, which we may call a *fiber*, that in a reference configuration $\mathscr{R}$ has the form of a long, thin cylindrical rod of circular cross section and diameter D. Suppose the motion is such that, in a cylindrical coordinate system with the Z-axis along the axis of the rod,

$$\theta = \Theta \qquad r = \nu(Z, t)R \qquad z = \tilde{z}(Z, t) \qquad (0 \leq R \leq D/2) \tag{16}$$

where θ, r, z are the coordinates at time t of the material point with coordinates Θ, R, Z, in $\mathscr{R}$. As the material is assumed incompressible, the motion is isochoric and hence

$$\nu = \lambda^{-1/2} \tag{17}$$

with λ the (axial) stretch, defined by

$$\lambda = \lambda(Z, t) = \partial\tilde{z}(Z, t)/\partial Z = \tilde{z}_Z(Z, t) \tag{18}$$

Note that, by (16) and (17), the function $\tilde{z}$ determines the motion. For the dynamical processes most frequently considered in the present theory, it can be assumed that the body has been in the configuration $\mathscr{R}$ for all times previous to $t = 0$, i.e., that for $\zeta < 0$, $\tilde{z}(Z, \zeta) = Z$, and hence $\nu(Z, \zeta) = \lambda(Z, \zeta) = 1$.

For a motion of the type (16) one can show, after some calculation, that the proper numbers $\xi_i^t(s)$ of the tensor $\mathbf{G}^t(s)$ ($s \geq 0$) obey the relations

$$\left.\begin{aligned}
\xi_1^t(s) &= \lambda_{(t)}(s)^2\left[1 + \frac{\omega_{(t)}(s)^2R^2}{4\lambda(t)^2\lambda(t-s)(\lambda_{(t)}(s)^3 - 1)}\right] + O(\omega_{(t)}(s)^4R^4)\\
\xi_2^t(s) &= \lambda_{(t)}(s)^{-1}\left[1 - \frac{\omega_{(t)}(s)^2R^2}{4\lambda(t)^2\lambda(t-s)(\lambda_{(t)}(s)^3 - 1)}\right] + O(\omega_{(t)}(s)^4R^4)\\
\xi_3^t(s) &= \lambda_{(t)}(s)^{-1}
\end{aligned}\right\} \tag{19}$$

in which

$$\lambda_{(t)}(s) = \lambda(t)/\lambda(t-s) \tag{20a}$$

$$\omega_{(t)}(s) = [\lambda_{(t)}(s)]_Z = \frac{\partial}{\partial Z}\left[\frac{\lambda(Z, t)}{\lambda(Z, t-s)}\right] \tag{20b}$$

The value $\psi(t)$ of the functional $\not{p}$ in eq. (7) is the density of the Helmholtz free energy per unit volume of fiber regarded as a three-dimensional fiber. The average of $\psi(t)$ over the cross section with axial location Z in $\mathcal{R}$, i.e.,

$$\Psi(Z, t) = \frac{4}{\pi D^2} \int_0^{2\pi} \int_0^{D/2} \psi(t, Z, R, \Theta) R dR d\Theta \tag{21}$$

is the linear density of the Helmholtz free energy at the material coordinate Z at time t (per unit of length in the reference configuration) for the fiber regarded as a unidimensional continuum; this linear density Ψ has been normalized by division by the cross-sectional area of the fiber in its unstrained configuration $\mathcal{R}$. By placing (19) into (13), one may show that

$$\Psi(t) = \int_0^\infty f(s, \lambda_{(t)}(s)) ds - \frac{1}{2} \int_0^\infty g(s, \lambda(t), \lambda(t-s)) \omega_{(t)}(s)^2 ds + O(UD^4) \tag{22}$$

where

$$U = \sum_{j=1}^{n} \int_0^\infty K_j(s)^4 \omega_{(t)}(s)^4 ds \tag{23}$$

and the functions f and g are related as follows to the functions K_j and h_j that characterize the three-dimensional material:

$$f(s, \xi) = \sum_{j=1}^{n} K_j(s) h_j(\xi^2, \xi^{-1}, \xi^{-1}) \tag{24}$$

$$g(s, \lambda(t), \lambda(t-s)) = \frac{-D^2}{32\lambda(t)\lambda(t-s)(\lambda_{(t)}(s)^3 - 1)} \frac{\partial}{\partial \lambda(t)} f\left(s, \frac{\lambda(t)}{\lambda(t-s)}\right) \tag{25}$$

It will be noticed that the function g in the second term (on the right) in (22) is proportional to D^2 and is determined by the function f which is independent of D and which governs the response of the fiber to homogeneous deformations. As

$$\omega_{(t)}(s) = \frac{\lambda(t-s)\lambda_Z(t) - \lambda(t)\lambda_Z(t-s)}{\lambda(t-s)^2} \tag{26}$$

the second term in (22) tells us that for inhomogeneous motions, when terms $O(D^2)$ are taken into account, Ψ is influenced by the history of the spatial gradient λ_Z of the stretch λ and the term of the lowest order containing the history of λ_Z is quadratic in that history.

Neglect of the term $O(UD^4)$ in (22) yields an explicit constitutive equation for the (normalized) linear density of free energy in a theory of fibers regarded as unidimensional continua. In such a theory the basic kinematical variable is $z = \tilde{z}(Z, t)$, and under the assumption that the resultant of the extrinsically applied forces per unit length, if it does not vanish, has a component $b(Z, t)$ only in the Z-direction, the law of balance of linear momentum becomes

$$T_Z + \rho b = \rho \ddot{z} \tag{27}$$

with $\ddot{z} = \partial^2 \tilde{z}(Z, t)/\partial t^2$, with ρ the mass density (which also equals the normalized linear mass density per unit of length in $\mathcal{R}$), and with T the normalized tension in the fiber, i.e., the total tensile force acting across a normal section divided by the cross-sectional area in $\mathcal{R}$. By a somewhat lengthy argument similar to that by which (11) is derived from (7), it is shown that the constitutive equation for T resulting from omission of the term $O(UD^4)$ in (22) is

$$T(t) = \int_0^\infty \frac{\partial}{\partial \lambda(t)} f\left(s, \frac{\lambda(t)}{\lambda(t-s)}\right) ds - \frac{1}{2} \int_0^\infty \frac{\partial}{\partial \lambda(t)} [g(s, \lambda(t), \lambda(t-s)) \omega_{(t)}(s)^2] ds + \frac{\partial}{\partial Z} \int_0^\infty \frac{g(s, \lambda(t), \lambda(t-s))}{\lambda(t-s)} \omega_{(t)}(s) ds \tag{28}$$

Moreover, one can show[2] that the unidimensional theory of nonhomogeneous axial motions of a viscoelastic fiber obtained by taking (28) as an exact expression for the tension is compatible with the second law of thermodynamics in that it yields a non-negative expression for the rate of internal dissipation, i.e., that part of the rate of decrease of Ψ that results from past changes in λ, λ_Z, and λ_{ZZ}.

AN ILLUSTRATIVE EXAMPLE

So as to have, for illustrative numerical calculation, an example of a three-dimensional material susceptible to cold drawing that is describable by constitutive relations of the type (7)–(11), we may let the number n be 3 and suppose that the functions H_j have the forms

$$\left.\begin{aligned} H_1(\mathbf{G}) &= \frac{\mu_1}{2}\,\mathrm{tr}(\mathbf{G} - \mathbf{1}) \\ H_2(\mathbf{G}) &= \frac{\mu_2}{4}\,\mathrm{tr}(\mathbf{G}^2 - \mathbf{1}) \\ H_3(\mathbf{G}) &= 2\mu_3\mathrm{tr}(4\mathbf{1} - [\mathbf{G} + 3\mathbf{1}]e^{-(\mathbf{G}-\mathbf{1})/4}) \end{aligned}\right\} \quad (29)$$

in which μ_1, μ_2, μ_3 are appropriate positive constants. By putting

$$\mu_1 = 0.05 \qquad \mu_2 = 0.002 \qquad \mu_3 = 0.35 \quad (30)$$

we obtain a viscoelastic material for which, as we shall explain below [after eq. (39)], the instantaneous (i.e., high-speed) stress response from a rest state in which $\mathbf{G}^t \equiv \mathbf{1}$, is the same as that for an illustrative elastic material studied in our earlier work.[9,10] The formulae (29) yield the following expressions for the functions h_j in the relations (13) and (24):

$$\left.\begin{aligned} h_1(\xi_1, \xi_2, \xi_3) &= \sum_{i=1}^{3} \frac{\mu_1}{2}(\xi_i - 1) \\ h_2(\xi_1, \xi_2, \xi_3) &= \sum_{i=1}^{3} \frac{\mu_2}{4}(\xi_i^2 - 1) \\ h_3(\xi_1, \xi_2, \xi_3) & \\ &= \sum_{i=1}^{3} 2\mu_3[4 - (\xi_i + 3)e^{-(\xi_i - 1)/4}] \end{aligned}\right\} \quad (31)$$

It is easily verified that the relations (14) and (15) hold here. The functions K_j can be written

$$K_j(s) = -d\mathsf{G}_j(s)/ds \quad (32)$$

where, by (9), each G_j is a relaxation function normalized so that

$$\mathsf{G}_j(0) = 1 \quad (33)$$

It is a consequence of this normalization that the small-strain high-speed moduli of the material are determined by the material parameters μ_j. [The values shown in (30) give $E(1) = 1.212$ and $\mu_s(1) = 0.404$, respectively, for the tensile modulus and shear modulus for rapid small deformations from the state in which $G^t \equiv \mathbf{1}$. A change in units of force is equivalent to multiplication of the μ_j by a single scale factor.] There is a broad class of semicrystalline polymers whose viscoelastic response can be described, approximately, by relaxation functions that vary, for large values of s (e.g., for $s > 1$, in seconds), as $(c + s)^{-\alpha}$, with α in the range $0.05 < \alpha < 0.2$. By putting

$$\mathsf{G}_1(s) = (1 + s)^{-0.12} \qquad \mathsf{G}_2(s) = (1 + s)^{-0.06}$$
$$\mathsf{G}_3(s) = (1 + s)^{-0.18} \quad (34)$$

i.e.,

$$\begin{aligned} K_1(s) &= 0.12(1 + s)^{-1.12} \\ K_2(s) &= 0.06(1 + s)^{-1.06} \\ K_3(s) &= 0.18(1 + s)^{-1.18} \end{aligned} \quad (35)$$

one can mimic, in a rough way, the rheological behavior of typical semicrystalline polymers susceptible to drawing. It is clear that the expressions (35) are compatible with (8).

When the deformation of a fiber is homogeneous, i.e., when λ in eq. (18) is independent of Z, the constitutive eq. (28) reduces to

$$T(t) = \int_0^\infty \frac{\partial}{\partial\lambda(t)} f\left(s, \frac{\lambda(t)}{\lambda(t-s)}\right) ds \quad (36)$$

To gain some insight into the implications of the relations (30)–(34) for homogeneous stretch, one considers a motion in which a previously unstrained fiber with length L is, at time $t = 0$, rapidly and homogeneously stretched to length $\lambda_0 L$, $\lambda_0 > 1$, and is held at fixed length for subsequent times. In such a motion the tension, $T(t)$, jumps to a positive value at $t = 0$ and then decays. In eq. (36) one has, for $t > 0$,

$$\lambda(t - s) = \begin{cases} \lambda_0, & 0 \le s \le t \\ 1, & s > t \end{cases} \quad (37)$$

and hence $T(t)$ is given by a function τ of λ_0 and t. On putting the expressions (24), (31), and (32) into (36), we find that for the history (37)

$$\begin{aligned} \tau(t, \lambda_0) &= \mu_1(\lambda_0 - \lambda_0^{-2})\mathsf{G}_1(t) + \mu_2(\lambda_0^3 - \lambda_0^{-3})\mathsf{G}_2(t) \\ &\quad + \mu_3[\lambda_0(\lambda_0^2 - 1)e^{-(\lambda_0^2-1)/4} \\ &\qquad + \lambda_0^{-2}(1 - \lambda_0^{-1})e^{(1-\lambda_0^{-1})/4}]\mathsf{G}_3(t) \end{aligned} \quad (38)$$

and it follows from (33) that the tension immediately after the jump in strain, $T(0+)$, is

$$T(0+) = \tau(0+, \lambda_0) = \mu_1(\lambda_0 - \lambda_0^{-2}) + \mu_2(\lambda_0^3 - \lambda_0^{-3}) + \mu_3\lambda_0(\lambda_0^2 - 1)e^{-(\lambda_0^2-1)/4} + \mu_3\lambda_0^{-2}(1 - \lambda_0^{-1})e^{(1-\lambda_0^{-1})/4} \quad (39)$$

The function $\lambda_0 \mapsto \tau(0+, \lambda_0)$, which describes the instantaneous response to rapid homogeneous stretch of a previously undeformed viscoelastic fiber, has here the same form as the function $\lambda \mapsto \tau(\lambda)$ shown in eq. (62a) of our study[9,10] of elastic fibers susceptible to cold drawing.

A graph of the function $\lambda_0 \mapsto \tau(t, \lambda_0)$ for fixed t is called a *stress-relaxation isochrone*. In Figure 1 we show such isochrones based on eq. (39), with the parameters μ_j as in (30) and the functions G_j as in (34). The topmost curve is a graph of $\lambda_0 \mapsto \tau(0+, \lambda_0)$. The curves are, in descending order, isochrones for times $t = 0+, 10^{-1}, 1, 10^1, 10^2, \ldots, 10^9$ seconds. The isochrones for $t = 0+$ and $t = 10^{-1}$ seconds are nearly indistinguishable. For this particular (hypothetical) material, when t exceeds 4×10^7 seconds ($\sim$ 16 months) the isochrones are monotone. For each isochrone corresponding to a shorter time, there is an interval (λ_a, λ_b) of values of λ_0 on which $\partial\tau(t, \lambda_0)/\partial\lambda_0$ is negative; for λ outside this interval, $\partial\tau(t, \lambda_0)/\partial\lambda_0$ is positive with $\tau(t, \lambda_0)$ increasing without bound as λ_0 increases above λ_b. This mathematical property is a characteristic of viscoelastic materials susceptible to cold drawing; it implies that under the right conditions homogeneous configurations will lose stability and either jumps in strain or neck formation, or both, will occur. The matter is treated in some detail in another paper[2] in which an analysis is made of the creep response to static loads, and the stability criterion of Coleman and Zapas[11] is shown to hold here. (For related criteria and reports of experimental observations of the relation between applied load and the time required for neck formation in high density polyethylene, see the papers of Crissman and Zapas.[12,13])

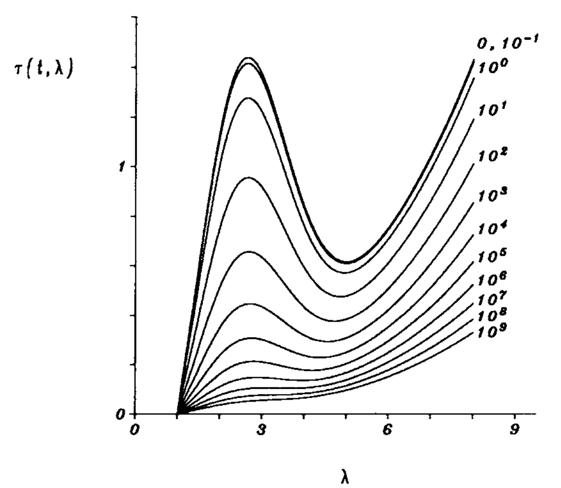

Figure 1 Stress-relaxation isochrones at indicated values of t for a fiber of a material obeying eq. (11) with H_j and K_j as in eqs. (29), (30), and (35).

We give below the results of numerical calculations showing that the present theory predicts that loss of stability can cause the formation of necks that closely resemble those observed. Of course, when doing calculations for motions that are not homogeneous i.e., for which λ varies with Z, the terms on the right in eq. (28) that are not seen in (36) cannot be neglected. The details of the numerical methods employed are given in Newman's doctoral dissertation.[14]

CREEP UNDER CONSTANT LOAD

A motion in which

$$T(t) = \begin{cases} 0, & t < 0 \\ T^\circ, & t > 0 \end{cases} \quad (40)$$

with T° a positive constant, is called *creep under dead load*, or, for short, *creep*.

In Figure 2 one sees calculated fiber profiles showing neck formation in creep. The numerical problem treated here is that of solving eq. (28) to obtain λ as a function of z and t under the assumption that $T(t)$ corresponds to creep under dead load. [The functions f and g in (28) are given by (24) and (25) with the functions h_j and K_j chosen to be those of the illustrative material whose stress-relaxation isochrones for homogeneous response are shown in Figure 1.] An area-reduction technique was employed to localize the place where neck formation might begin, i.e., the fiber was weakened by a slight reduction of the initial cross-sectional area in the region whose material boundary is indicated in the figure with vertical dashes. In this case T°, the normalized tension (per unit of initial area outside of the thinned region) was 0.6 and gave rise to a value of 0.741 for the tensile stress per unit initial area at the center of the thinned region. Calculations based on eq. (36) show that, for a perfectly homogeneous fiber of the viscoelastic material of the present numerical study, when the dead load per unit initial

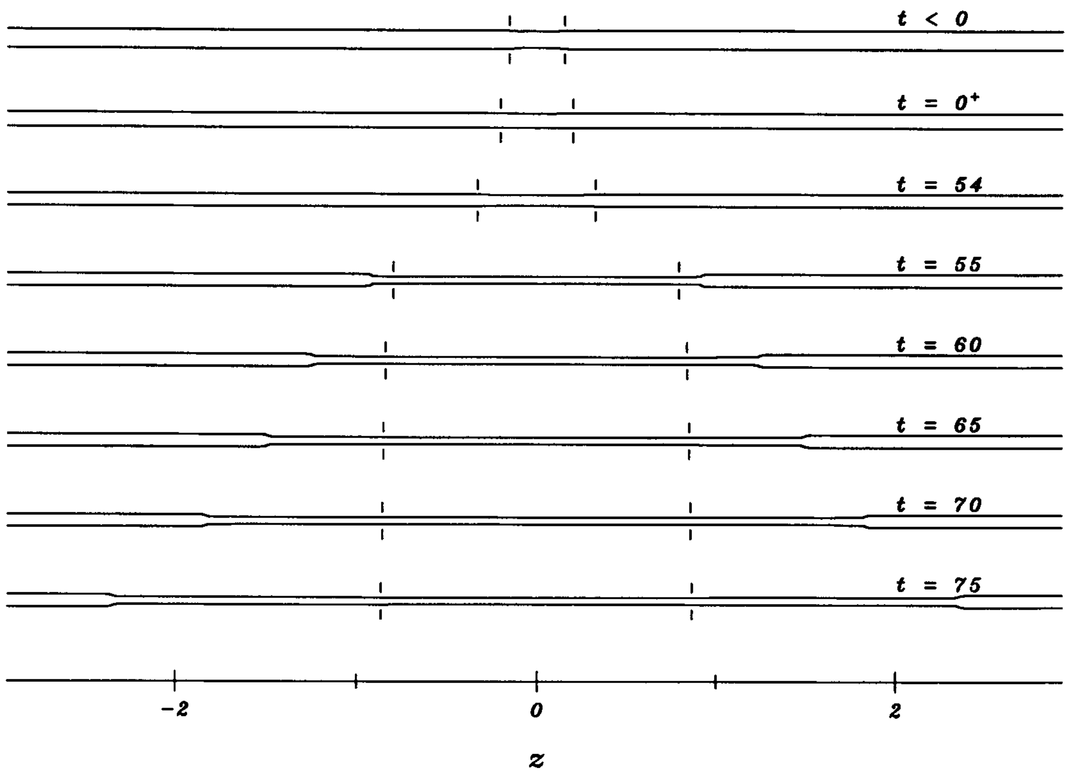

Figure 2 Calculated fiber profiles showing the formation and growth of a neck in creep under a dead load. The fiber, which is assumed to be composed of a material with the stress-relaxation isochrones seen in Figure 1, is weakened by a small reduction of initial diameter in the region whose material boundaries are indicated with vertical marks.

area equals 0.741, homogeneous creep becomes unstable when $t = 54.5$ seconds, and, in accord with the theory of Coleman and Zapas,[11] this instability, which results in a jump in stretch, occurs precisely at the moment of vanishing of the instantaneous (i.e., high-speed) tensile modulus.

As Figure 2 shows, in the present case neck formation initiates with a jump in stretch occurring in the thinned region at a time $t_{\#}$ between 54 and 55 seconds. For times $t > t_{\#}$, the fiber contains a region, the *neck*, in which pronounced stretching has occurred. The neck increases in length, at first slowly, during an "incubation period", and then rapidly. In the case presented here, after 82 seconds the length of the neck is increasing (and accelerating) so rapidly as to render not feasible further calculation of the stretch field λ by direct solution of eq. (28). Our numerical study has indicated that, for a broad range of initial weakening by area reduction, there is a time $\bar{t}$ (in the present case $\bar{t}$ is about 77 seconds) such that the rate of advance of the neck at times $t > \bar{t}$ is independent of the amount of weakening: within limits, the more an otherwise homogeneous fiber is weakened by area reduction in a short region, the sooner a neck starts to form in that region, but the longer the neck must "incubate" before its edges advance rapidly into homogeneously deformed material.

One can show that there is a time, t_∞, at which neck growth occurs at a rate that is essentially infinite on the internal time scales of the material; under the conditions that are appropriate to Figure 2, $t_\infty = 105$ seconds. The shape of the fiber near the edges of the neck at time t_∞ can be calculated using instantaneous response functions for T and the methods which were developed to calculate the shape of a "fully developed draw" in a theory[15,16] of neck formation in elastic materials susceptible to cold drawing.

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